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GENERALIZED YOUDEN DESIGNS: CONSTRUCTION AND TABLES.(U)
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Generalized Youden Designs

Construction and Tables[1]

by

Arlene Ash
Department of Mathematics
University of Illinois at Chicago 409 950

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SUMMARY

Generalized Youden Designs: Construction and Tables

by

Arlene Ash
University of Illinois at Chicago
and
Boston University

Generalized Youden Designs are generalizations of the class of two-way balanced block designs which include Latin squares and Youden squares. They are used for the same purposes and in the same way that these classical designs are used, and satisfy most of the most common criteria of design optimality.

We explicitly display or give detailed instructions for constructing all these designs within a practical range: when v, the number of treatments, is ≤ 25; and b_1 and b_2, the dimensions of the design array, are each ≤ 50.

ABBREVIATED TITLE: Generalized Youden Designs.

FOOTNOTES

[1] This paper is part of the author's doctoral dissertation written at the University of Illinois at Chicago Circle. Research sponsored in part by Dartmouth College and Grant AFOSR-76-350A.

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[2] GYD(v,b_1,b_2) and GYD(v',b_1',b_2') are in the same family if $v = v'$ and b_i is congruent to b_i' mod v, for $i = 1,2$.

[3] The rows of a B_0 = GYD(22,22,7) would be a BIB(22,22,7) which cannot be constructed. See for example (Collens, 1976). A design with twice as many blocks, the smallest possible in this family, is listed instead.

[4] No small design in this family exists, because of the non-existence of required BIB(15,21,5). See for example (Collens, 1976).

Generalized Youden Designs: Construction and Tables[1]

by

Arlene Ash
University of Illinois at Chicago
and
Boston University

1. Introduction. A Generalized Youden Design (GYD) is a $b_1 \times b_2$ array on the treatments $\{1,\ldots,v\}$ denoted GYD(v,b_1,b_2), whose induced row and column designs satisfy certain technical conditions of balance. It is a natural generalization of several well-known highly balanced two-way design structures, such as the Latin square and the Youden square. Indeed, when $b_1 = v$ and $b_2 \le v$, a GYD(v,b_1,b_2) is exactly one of these designs. The GYD definition extends the collection of design settings in which two-way balance may be achieved, to include values of b_1 and b_2 which may be greater than v.

GYD's are used for comparing the performance of v treatments in the presence of two-way heterogeneity (in, respectively, b_1 and b_2 blocks) under the standard linear model with no interactions. For such purposes GYD's are optimal under several of the most common criteria of optimality [6]. This paper makes these statistically good designs available by listing (or giving explicit directions for constructing) those GYD's within a practical range: $v \le 25$; $b_1,b_2 \le 50$.

Associated with each design is a $v \times v$ matrix called the information matrix of the design (for the treatment means). Usually the goodness of a design is measured as some function of this matrix. Since non-isomorphic GYD's with the same parameters (v,b_1,b_2) have identical information matrices, we indicate only one possible construction for each parameter set where construction can be achieved. In the range specified there is only one

parameter set ($v = 25$, $b_1 = b_2 = 40$) where neither nonexistence of the GYD, nor any construction is known. We have, however, provided a 40×40 design, P18, the combined set of whose rows and columns form a balanced 80-block design. This makes it a Pseudo Youden Design (PYD) as defined by Cheng (1978) and gives it the same information matrix and optimality properties as the missing GYD.

Previous work by Seiden, Ruiz and Wu (1974, 1978) has provided explicit information for constructing many designs where v is a power of a prime. Theorems by Kiefer (1975) combined with lists of BIB's (Hanani, 1975 or Collens, 1976 for example) guarantee the existence of some other designs in our listing. Designs P18 and N3, 6, 10, 11, 12, 14 and 16 are apparently new.

Special acknowledgements are due to Ching-Shiu Cheng who pointed out that a BIB(25,80,15) could be substituted for two BIB(25,40,15)'s in making a PYD surrogate for GYD(25,40,40), and to Haim Hanani who provided the eighty block design which was used in making that construction. Thanks also to R. J. Collens who helped locate a needed BIB design from the updated listing he maintains on the University of Manitoba computer.

It still seems of some theoretical interest to know whether or not any GYD(25,40,40) exists.

2. Definitions and notation. Throughout this paper we will let $b_i = m_i v + c_i$, $0 \le c_i < v$, $i = 1,2$.

Definition. A $GYD(v,b_1,b_2)$ is a $b_1 \times b_2$ array with entries from $V = \{1,\ldots,v\}$ such that

(a) Every element of V occurs either m_1 or $m_1 + 1$ times in each column and either m_2 or $m_2 + 1$ times in each row; and

(b) Every two distinct elements of V occur together in the same row the same number of times; and similarly for columns.

That is, the rows of a GYD form a so-called balanced block design, $BBD(v,b=b_1,k=b_2)$; and the columns form a $BBD(v,b=b_2,k=b_1)$. The existence of BBD's with the given parameters is equivalent to the existence of balanced incomplete block designs $BIB(v,b=b_1,k=c_2)$ and $BIB(v,b=b_2,k=c_1)$. Well-known parameter conditions are necessary for the existence of BIB's, and consequently, for a fixed (v,c_1,c_2), $GYD(v,b_1=m_1v+c_1,b_2=m_2v+c_2)$ can only exist when v divides c_1c_2 and the b_i, in addition to being congruent to c_i mod v, satisfy these BIB-related conditions.

3. <u>Construction</u>. GYD construction is trivial when v divides both b_1 and b_2, and essentially equivalent to the construction of a related BIB when v divides just one of b_1 or b_2 but not both. It is new and interesting when neither b_1 nor b_2 is a multiple of v. We distinguish these three design settings as, respectively, the doubly regular, (simply) regular and non-regular cases.

3.1 <u>The doubly regular case</u>. $GYD(v, b_1 = m_1 v, b_2 = m_2 v)$ can be constructed by juxtaposing any m_1 by m_2 collection of Latin squares. For example, let the (i,j)th entry of the array be v if $i + j \equiv 1 \bmod v$ and $(i + j - 1) \bmod v$ otherwise.

3.2 <u>The (simply) regular case</u>. Transposing the design if necessary we may assume that $c_1 = 0$. We may construct $GYD(v, b_1 = m_1 v, b_2 = m_2 v + c_2)$ by adjoining a doubly regular $GYD(v, m_1 v, m_2 v)$ to a regular $GYD(v, m_1 v, c_2)$.

For every v and $0 < c_2 < v$, there is a $b_0 = m_0 v$, such that if regular $B_1 = GYD(v, b_1 = m_1 v, c_2)$ exists then b_1 is a multiple of b_0. If $B_0 = GYD(v, b_0, c_2)$ exists, then B_1 can be constructed by stacking b_1/b_0 copies of B_0. $B_0 = B_0(v, c_2)$ is called a <u>core design for the (v,c_2)-family</u> of regular designs. We list instructions for finding B_0:

3.2.1. If $c_2 = 1$, B_0 may be taken as the $(v \times 1)$ column array $[1,2,\ldots,v]'$, where the "prime" indicates transpose.

3.2.2. If $c_2 = 2$, B_0 will be a $v(v-1) \times 2$ array for v even, and a $[v(v-1)/2] \times 2$ array for odd v. In either case, construct the first column of B_0 by stacking copies of column vector $D = [1,2,\ldots,v]'$; and the second column by cycling the entries of D : i.e., first use $[2,3,\ldots,v,1]'$, then $[3,4,\ldots,v,1,2]'$, etc. until the second column is filled. E.g. for $v = 5$ we get

$$B_0 = B_0(5,2) = GYD(5,10,2) = \left[\begin{array}{c|c} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \\ 5 & 1 \\ \hline 1 & 3 \\ 2 & 4 \\ 3 & 5 \\ 4 & 1 \\ 5 & 2 \end{array}\right] .$$

3.2.3. If $3 \le c_2 \le v/2$, the construction of B_0 is more complicated. For $v \le 25$ and b_0 no bigger than 50, the design B_0 will be found in the regular design tables through the parameters (v,c_2).

3.2.4. For $c_2 > v/2$, there is a three-step procedure. First, locate B_0', the core design for the $(v,c_2'=v-c_2)$ family, as above. Then, construct comp(B_0'), a row design complementary to B_0' viewed only as a row design. That is, put $i \in \{1,\ldots,v\}$ in the j^{th} row of the complementary design when i is not in the j^{th} row of the original. This new design is a BIB(v,b_1,c_2), but the array may require rearrangement of the entries within rows so that the columns are also balanced. This can always be done (Agrawal, 1966), and for small designs it can be done rapidly by eye. Sometimes, if B_0' has been constructed with a discernible column pattern this facilitates writing B_0

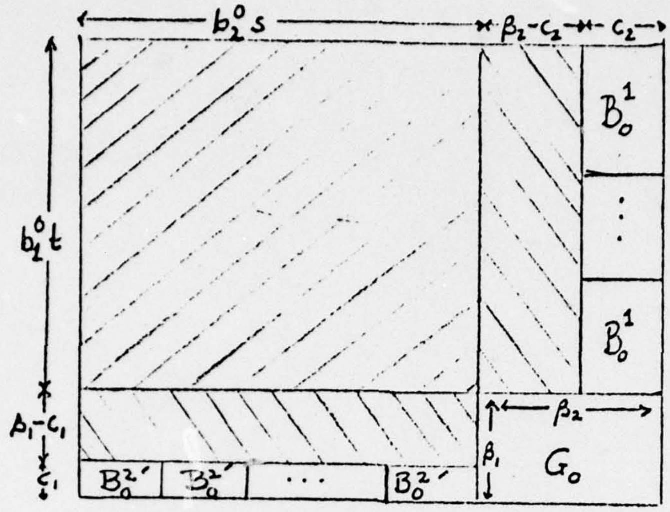
directly as a GYD. E.g., for GYD(7,21,18) we need GYD(7,21,4). Since $4 > 7/2$, we seek B_0', the core design for the ($v=7$, $c_2'=3$) family and find it, in section 4.2, as R2:

$$B_0' = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \\ 4 & 5 & 7 \\ 5 & 6 & 1 \\ 6 & 7 & 2 \\ 7 & 1 & 3 \end{bmatrix} \quad ; \quad \mathrm{comp}(B_0') = \begin{bmatrix} 3 & 5 & 6 & 7 \\ 1 & 4 & 6 & 7 \\ 1 & 2 & 5 & 7 \\ 1 & 2 & 3 & 6 \\ 2 & 3 & 4 & 7 \\ 1 & 3 & 4 & 5 \\ 2 & 4 & 5 & 6 \end{bmatrix} \quad \text{or } \mathrm{comp}(B_0') = \begin{bmatrix} 3 & 5 & 6 & 7 \\ 4 & 6 & 7 & 1 \\ 5 & 7 & 1 & 2 \\ 6 & 1 & 2 & 3 \\ 7 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 2 & 4 & 5 & 6 \end{bmatrix}$$

A 21×14 doubly regular design, patched to a stack of three copies of the second version of $\mathrm{comp}(B_0')$, makes the desired GYD.

If "eyeball" methods fail to produce a rearrangement of the BIB which is column-balanced the appendix contains a definitive algorithm (called SHUFFLE) for turning a BIB with b a multiple of v into a GYD.

3.3. <u>The Nonregular Case</u>. GYD(v,b_1,b_2)'s exist for $b_i \equiv c_i \bmod v$, only when $v|c_1c_2$. For such a triple (v,c_1,c_2) there are integers β_1, β_2, b_1^0 and b_2^0 such that any GYD(v,b_1,b_2) with $b_i \equiv c_i \bmod v$ has $b_1 = \beta_1 + b_1^0 s$ and $b_2 = \beta_2 + b_2^0 t$, for some $s,t \in \{0,1,2,\ldots\}$. Here the β_i's are minimal possible dimensions in the (v,c_1,c_2)-family; and the b_i^0's are multiples of v for which BIB's in b_0^i blocks of size c_i can exist. $G_0 = \mathrm{GYD}(v,\beta_1,\beta_2)$, if it exists, is called a <u>core design</u> for the family. If, further, regular designs $B_0^1 = \mathrm{GYD}(v,b_1^0,c_2)$ and $B_0^2 = \mathrm{GYD}(v,b_2^0,c_1)$ exist, then any GYD in this family can be constructed. One method is indicated in the diagram below, where the hatched regions are doubly regular designs.



Since nonregular designs cannot readily be assembled from the associated row and column designs, explicit constructions for the small core designs are given in 4.3. The parameters of the associated B_0^i designs are indicated next to those of G_0 in the listing which precedes the constructions. Most of these, however, have too many blocks to be included in the tables of regular designs.

4. <u>Guide to the tables</u>. The tables contain the building blocks for $GYD(v,b_1,b_2)$'s with $v \leq 25$ and b_1,b_2 both ≤ 50. We refer to such designs as "small". In order to determine whether or not a small GYD exists and to find its construction if it does, perform the numbered steps sequentially until they lead to an exit.

The exits are coded by

□□ $GYD(v,b_1,b_2)$ does not exist, nor does any GYD in the same family[2] exist.

□ $GYD(v,b_1,b_2)$ with b_1 and b_2 both ≤ 50 does not exist although larger GYD's in the same family will.

⇨ Proceed elsewhere, as directed.

The procedure is definitive for $v \leq 25$ and b_1,b_2 both ≤ 50, although a Pseudo Youden Design (see the Introduction) has been offered instead of GYD(25,40,40) in the one place where we have not been able to achieve the more rigid construction. Even if b_1 or b_2 is bigger than 50, if $v \leq 25$ a construction may result if the GYD sought is in the same family as another design which is in the range of the tables. For $v > 25$, the procedure is only useful for determining when no design can exist.

4.1. <u>Procedure for resolving a $GYD(v,b_1,b_2)$ construction problem.</u>

1) Compute c_i, the residue of b_i mod v, for i = 1,2.
2) If $v \nmid c_1 c_2$, then no design exists. □□
3) If $c_1 = c_2 = 0$, the design construction is doubly regular and can be achieved. See 3.1. ⇨

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[2] $GYD(v,b_1,b_2)$ and $GYD(v',b_1',b_2')$ are in the same family if $v = v'$ and b_i is congruent to b_i' mod v, for i = 1,2.

4) If necessary relabel b_1 and b_2 so that $c_1 \le c_2$.

5) If $\frac{b_1c_2}{v}$ and $\frac{b_1c_2(c_2-1)}{v-1}$ are not both integers then no design exists. ◫

6) If $c_1 = 0$, then the construction problem is regular. In this case do the following:

   a) If $c_2 = 1$ or 2 the design construction is trivial. See 3.2.1 or 3.2.2, respectively. ⇨

   b) Compute $h_2 = \text{g.c.d.}(v-1, c_2(c_2-1))$.

   c) If $v(\frac{v-1}{h_2}) > 50$, no small design exists. □

   d) If $3 \le c_2 \le v/2$, look for the design in 4.2. ⇨

   e) If $c_2 > v/2$, see 3.2.4. ⇨

7) If $\frac{b_2c_1}{v}$ and $\frac{b_2c_1(c_1-1)}{v-1}$ are not both integers, then no design exists. ◫

8) Compute $g_i = \text{g.c.d.}(v, c_i)$ and $h_i = \text{g.c.d.}(v-1, c_i(c_i-1))$, for $i = 1,2$.

9) If either $\frac{v(v-1)}{g_2h_2}$ or $\frac{v(v-1)}{g_1h_1}$ is > 50, no small design exists. □

10) If $v = 25$ and $b_1 = b_2 = 40$, it is not known whether a GYD exists. An equivalent design PYD(25,40,40) is listed in 4.3 as design P18. ⇨

11) Look for the design in 4.3. ⇨

4.2. Listings for regular designs. ($v \le 25$; $c_1 = 0$; $c_2 \ne 0$).

For $\text{GYD}(v, b_1=m_1v, b_2=m_2v+c_2)$ to exist b_1 must be a multiple of

$$b_0 = b_0(v,c_2) = \frac{v(v-1)}{\text{g.c.d.}(v-1, c_2(c_2-1))},$$

where g.c.d. denotes greatest common divisor.

For $v \le 25$, with $3 \le c_2 \le v/2$, if $b_0 \le 50$, this table displays $B_0 = GYD(v,b_0,c_2)$, from which G can be made by stacking b_1/b_0 copies of B_0 and adjoining an $m_1v \times m_2v$ doubly regular design (see 3.1). For one parameter set in this range, $v = 22$ and $c_2 = 7$, B_0 does not exist, and a design with $2b_0 = 44$ blocks is given instead.

For $v \le 25$ and $c_2 > v/2$, B_0 can be constructed as the complement of a (v,c_2')-design, where $c_2' = v - c_2$. (See 3.2.3.)

For any v and $c_2 = 1$ or 2, see sections 3.2.1 or 3.2.2 respectively.

Listing Name	Family (v,c_2)	Dimensions of B_0 $GYD(v,b_0,c_2)$	Listing Name	Family (v,c_2)	Dimensions of B_0 $GYD(v,b_0,c_2)$
R1	(6,3)	(6,30,3)	R13	(16,5)	(16,48,5)
R2	(7,3)	(7,7,3)	R14	(16,6)	(16,16,6)
R3	(9,3)	(9,36,3)	R15	(17,8)	(17,34,8)
R4	(9,4)	(9,18,4)	R16	(19,9)	(19,19,9)
R5	(10,3)	(10,30,3)	R17	(21,5)	(21,21,5)
R6	(10,4)	(10,30,4)	R18	(21,6)	(21,42,6)
R7	(11,5)	(11,11,5)	R19	(21,10)	(21,42,6)
R8	(13,3)	(13,26,3)	R20	(22,7)	(22,44,7)[3]
R9	(13,4)	(13,13,4)	R21	(23,11)	(23,23,11)
R10	(13,5)	(13,39,5)	R22	(25,4)	(25,50,4)
R11	(13,6)	(13,26,6)	R23	(25,9)	(25,25,9)
R12	(15,7)	(15,15,7)	R24	(25,12)	(25,50,12)

Table 1: Small, core regular designs.

[3] The rows of $B_0 = GYD(22,22,7)$ would be a BIB(22,22,7), which cannot be constructed. See for example (Collens, 1976). A design with twice as many blocks, the smallest possible in this family, is listed instead.

R1 IS A GYD(6 , 30 , 3)

1	3	5
5	2	6
5	3	6
6	1	4
5	4	2
4	6	3
4	1	5
6	1	2
1	2	3
3	2	4
1	3	5
2	5	6
6	5	3
6	4	1
2	4	5
3	6	4
4	1	5
2	6	1
3	2	1
3	2	4
1	5	3
2	5	6
3	5	6
1	6	4
5	4	2
4	6	2
5	4	1
6	1	2
2	3	1
4	3	2

R2 IS A GYD(7 , 7 , 3)

2	1	4
3	2	5
4	6	3
5	4	7
1	5	6
6	7	2
7	3	1

R3 IS A GYD(9 , 36 , 3)

2	1	3
4	5	6
7	8	9
4	1	7
8	5	2
3	6	9
5	9	1
2	7	6
8	3	4
8	6	1
9	2	4
7	5	3
1	3	2
4	6	5
9	8	7
1	4	7
8	5	2
6	9	3
5	9	1
7	2	6
3	8	4
6	8	1
2	4	9
5	3	7
1	3	2
4	6	5
9	7	8
1	7	4
5	2	8
6	9	3
9	1	5
7	2	6
3	4	8
6	1	8
2	4	9
3	7	5

R4 IS A GYD(9 , 18 , 4)

2	4	6	8
4	5	7	9
1	3	8	9
2	3	4	7
1	6	7	9
1	2	5	8
3	6	7	8
1	4	5	6
2	3	5	9
1	3	5	7
1	2	3	6
2	5	6	7
5	6	8	9
3	4	5	8
3	4	6	9
1	2	4	9
2	7	8	9
1	4	7	8

R5 IS A GYD(10 , 30 , 3)

1	2	9
10	3	2
3	6	4
4	7	5
8	5	1
5	3	6
7	4	1
2	5	8
9	1	3
2	4	10
1	2	6
2	3	7
4	8	3
4	5	9
10	1	5
6	9	4
5	7	10
1	6	8
9	2	7
8	10	3
6	10	9
6	10	7
7	8	6
7	9	8
9	8	10
3	9	5
10	1	4
5	6	2
3	7	1
8	4	2

R6 IS A GYD(10 , 30 , 4)

2	5	9	3
10	6	1	3
7	8	1	2
4	3	1	2
6	9	7	1
2	7	5	10
5	3	6	8
4	5	6	7
1	8	10	5
2	6	8	9
9	3	10	7
9	8	4	10
1	4	5	9
10	2	4	6
7	4	8	3
5	9	3	2
6	10	3	1
7	1	2	8
3	4	2	1
9	1	7	6
5	10	2	7
3	5	6	8
4	6	7	5
8	1	10	5
6	2	8	9
3	7	9	10
8	10	9	4
1	9	5	4
10	2	4	6
8	7	3	4

R7 IS A GYD(11 , 11 , 5)

5	3	4	1	9
2	6	10	5	4
3	11	5	6	7
8	1	7	4	6
7	9	8	2	5
9	10	6	8	3
10	4	9	7	11
11	5	1	10	8
6	2	11	9	1
1	7	2	3	10
4	8	3	11	2

R8 IS A GYD(13 , 26 , 3)

9	3	1
10	2	4
5	11	3
6	4	12
13	5	7
8	6	1
2	7	9
3	8	10
4	9	11
10	12	5
11	6	13
1	7	12
8	2	13
6	5	2
7	3	6
7	8	4
5	9	8
9	10	6
11	10	7
12	11	8
12	13	9
13	1	10
2	1	11
3	12	2
4	13	3
1	4	5

R9 IS A GYD(13 , 13 , 4)

1	10	3	2
4	5	6	10
7	8	10	9
11	1	4	7
6	2	11	5
6	3	9	11
12	9	5	1
2	12	7	6
3	4	8	12
13	6	1	8
9	13	2	4
5	7	13	3
10	11	12	13

R10 IS A GYD(13 , 39 , 5)

5	1	12	13	8
6	13	2	9	1
7	2	1	3	10
3	8	4	11	2
4	12	5	3	9
6	10	13	5	4
7	6	11	1	5
8	7	2	12	6
7	3	9	13	8
8	4	10	1	9
11	2	9	10	5
11	12	3	10	6
12	7	13	4	11
10	11	3	2	13
12	11	4	1	3
12	5	4	2	13
5	3	6	13	1
2	4	1	7	6
2	3	7	8	5
4	8	6	9	3
10	5	9	7	4
10	11	5	6	8
6	9	11	12	7
13	8	10	12	7
9	1	8	11	13
9	1	10	2	12
13	6	7	4	9
1	7	5	8	10
11	9	8	6	2
3	9	12	10	7
8	10	13	11	4
9	12	11	5	1
2	13	12	6	10
3	13	1	7	11
1	4	2	8	12
13	5	3	9	2
1	10	6	4	3
4	2	7	5	11
5	6	8	3	12

R11 IS A GYD(13 , 26 , 6)

10	3	4	9	12	1
2	11	5	4	13	10
6	1	12	5	3	11
13	12	4	2	6	7
7	13	5	3	8	1
8	9	1	2	4	6
3	9	2	10	7	5
4	8	10	11	6	3
5	7	12	11	9	4
13	6	8	10	5	12
1	6	13	9	11	7
1	2	7	8	10	12
9	2	11	8	13	3
11	5	6	7	2	8
12	3	6	7	8	9
8	13	10	4	7	9
11	10	9	5	1	8
9	11	2	6	12	10
7	12	13	3	10	1
12	8	11	13	1	4
2	5	1	12	9	13
6	10	3	1	2	13
4	7	3	1	11	2
3	4	8	12	5	2
5	4	9	13	3	6
10	1	7	6	4	5

K12 IS A GYD(15 , 15 , 7)

1	13	11	10	12	14	15
7	15	2	14	9	8	13
6	3	8	9	15	12	11
4	10	12	6	14	9	7
5	7	10	8	13	11	6
14	4	6	15	3	13	5
2	11	7	4	5	15	12
3	12	14	5	8	2	10
13	9	3	2	11	10	4
15	5	9	1	10	4	8
11	14	5	3	7	1	9
12	8	1	13	4	7	3
9	6	13	12	1	5	2
8	1	4	11	2	6	14
10	2	15	7	6	3	1

K13 IS A GYD(16 , 48 , 5)

5	3	6	9	12
16	4	8	2	1
12	10	11	7	15
1	5	14	13	16
9	14	5	15	7
2	11	10	16	13
7	8	4	12	9
13	6	3	1	2
4	16	7	11	5
3	15	13	14	10
5	1	9	6	4
10	12	2	8	3
11	2	12	4	6
14	9	1	3	8
6	13	15	5	11
8	7	16	10	14
1	4	7	10	13
12	3	13	5	7
16	5	9	3	2
15	10	2	4	9
13	11	12	8	1
7	14	1	6	12
2	6	15	14	16
9	8	16	11	15
10	15	6	1	8
5	16	8	12	6
3	12	11	16	14
4	1	14	15	11
8	9	5	13	10
6	2	10	7	5
14	7	4	2	3
11	13	3	9	4
2	5	8	11	14
9	10	6	14	11
13	4	14	6	8
7	3	11	8	6
16	6	10	4	3
15	8	5	3	4
1	11	3	5	10
12	14	4	10	5
14	12	13	9	2
11	1	7	2	9
8	15	2	7	13
6	16	9	13	7
3	7	1	15	16
4	13	12	16	15
10	9	16	12	1
5	2	15	1	12

K14 IS A GYD(16 , 16 , 6)

5	2	9	3	4	13
14	1	6	4	3	10
11	15	4	2	1	7
8	12	3	1	2	16
9	6	8	7	13	1
2	5	10	8	7	14
15	3	5	6	8	11
12	16	7	5	6	4
13	10	1	11	12	5
6	9	14	12	11	2
3	7	15	10	9	12
16	4	11	9	10	8
1	14	16	15	5	9
10	13	2	16	15	6
7	11	13	14	16	3
4	8	12	13	14	15

K15 IS A GYD(17 , 34 , 8)

1	4	9	16	8	2	15	13
2	5	10	17	9	3	16	14
3	6	11	1	10	4	17	15
4	7	12	2	11	5	1	16
5	8	13	3	12	6	2	17
6	9	14	4	13	7	3	1
7	10	15	5	14	8	4	2
8	11	16	6	15	9	5	3
9	12	17	7	16	10	6	4
10	13	1	8	17	11	7	5
11	14	2	9	1	12	8	6
12	15	3	10	2	13	9	7
13	16	4	11	3	14	10	8
14	17	5	12	4	15	11	9
15	1	6	13	5	16	12	10
16	2	7	14	6	17	13	11
17	3	8	15	7	1	14	12
3	5	6	7	10	11	12	14
4	6	7	8	11	12	13	15
5	7	8	9	12	13	14	16
6	8	9	10	13	14	15	17
7	9	10	11	14	15	16	1
8	10	11	12	15	16	17	2
9	11	12	13	16	17	1	3
10	12	13	14	17	1	2	4
11	13	14	15	1	2	3	5
12	14	15	16	2	3	4	6
13	15	16	17	3	4	5	7
14	16	17	1	4	5	6	8
15	17	1	2	5	6	7	9
16	1	2	3	6	7	8	10
17	2	3	4	7	8	9	11
1	3	4	5	8	9	10	12
2	4	5	6	9	10	11	13

K16 IS A GYD(19 , 19 , 9)

6	7	5	9	4	1	11	17	16
7	2	18	8	5	6	12	10	17
9	6	8	19	18	7	3	11	13
14	12	10	4	19	9	1	7	8
5	11	1	2	13	15	9	8	10
3	14	2	16	9	11	10	12	6
10	4	13	17	3	12	7	15	11
18	5	11	14	12	8	16	13	4
17	19	6	12	15	13	5	9	14
1	18	14	13	16	10	15	6	7
8	17	7	15	11	16	14	19	2
16	9	12	3	17	18	8	1	15
13	16	4	18	10	17	19	2	9
11	10	17	1	14	19	18	5	3
2	15	19	11	1	4	6	18	12
12	1	16	7	2	5	13	3	19
4	13	3	6	8	2	17	14	1
15	3	9	5	7	14	2	4	18
19	8	15	10	6	3	4	16	5

K17 IS A GYD(21 , 21 , 5)

4	3	1	2	17
8	5	17	7	6
11	17	10	12	9
14	13	16	17	15
18	9	13	5	1
2	10	14	6	18
15	11	3	18	7
12	16	18	4	8
9	19	7	16	2
13	12	6	3	19
5	14	4	19	11
19	15	8	1	10
20	8	2	11	13
16	20	5	10	3
7	1	20	14	12
6	4	15	9	20
21	2	12	15	5
3	21	9	8	14
10	7	21	13	4
1	6	11	21	16
17	18	19	20	21

K18 IS A GYD(21 , 42 , 6)

7	5	8	11	16	17
1	6	9	12	17	18
2	7	10	13	18	19
3	1	11	14	19	20
4	2	12	8	20	21
5	3	13	9	21	15
6	4	14	10	15	16
7	1	3	14	8	10
1	2	4	8	9	11
2	3	5	9	10	12
3	4	6	10	11	13
4	5	7	11	12	14
5	6	1	12	13	8
6	7	2	13	14	9
14	12	15	18	2	3
8	13	16	19	3	4
9	14	17	20	4	5
10	8	18	21	5	6
11	9	19	15	6	7
12	10	20	16	7	1
13	11	21	17	1	2
14	8	10	21	15	17
8	9	11	15	16	18
9	10	12	16	17	19
10	11	13	17	18	20
11	12	14	18	19	21
12	13	8	19	20	15
13	14	9	20	21	16
21	19	1	4	9	10
15	20	2	5	10	11
16	21	3	6	11	12
17	15	4	7	12	13
18	16	5	1	13	14
19	17	6	2	14	8
20	18	7	3	8	9
21	15	17	7	1	3
15	16	18	1	2	4
16	17	19	2	3	5
17	18	20	3	4	6
18	19	21	4	5	7
19	20	15	5	6	1
20	21	16	6	7	2

K19 IS A GYD(21 , 42 , 10)

1	4	8	9	10	12	15	16	17
2	5	9	10	11	13	16	17	18
3	6	10	11	12	14	17	18	19
4	7	11	12	13	8	18	19	20
5	1	12	13	14	9	19	20	21
6	2	13	14	8	10	20	21	15
7	3	14	8	9	11	21	15	16
14	10	21	15	16	18	7	1	2
8	11	15	16	17	19	1	2	3
9	12	16	17	18	20	2	3	4
10	13	17	18	19	21	3	4	5
11	14	18	19	20	15	4	5	6
12	8	19	20	21	16	5	6	7
13	9	20	21	15	17	6	7	1
20	16	6	7	1	3	13	14	8
21	17	7	1	2	4	14	8	9
15	18	1	2	3	5	8	9	10
16	19	2	3	4	6	9	10	11
17	20	3	4	5	7	10	11	12
18	21	4	5	6	1	11	12	13
19	15	5	6	7	2	12	13	14
4	5	8	10	13	15	16	17	21
5	6	9	11	14	16	17	18	15
6	7	10	12	8	17	18	19	16
7	1	11	13	9	18	19	20	17
1	2	12	14	10	19	20	21	18
2	3	13	8	11	20	21	15	19
3	4	14	9	12	21	15	16	20
10	11	21	16	19	7	1	2	6
11	12	15	17	20	1	2	3	7
12	13	16	18	21	2	3	4	1
13	14	17	19	15	3	4	5	2
14	8	18	20	16	4	5	6	3
8	9	19	21	17	5	6	7	4
9	10	20	15	18	6	7	1	5
16	17	6	1	4	13	14	8	12
17	18	7	2	5	14	8	9	13
18	19	1	3	6	8	9	10	14
19	20	2	4	7	9	10	11	8
20	21	3	5	1	10	11	12	9
21	15	4	6	2	11	12	13	10
15	16	5	7	3	12	13	14	11

R220 IS A GYD(22 , 44 , 7)

22	1	2	6	12	15	20
1	2	3	7	13	16	21
2	3	4	8	14	17	22
3	4	5	9	15	18	1
4	5	6	10	16	19	2
5	6	7	11	17	20	3
6	7	8	12	18	21	4
7	8	9	13	19	22	5
8	9	10	14	20	1	6
9	10	11	15	21	2	7
10	11	12	16	22	3	8
11	12	13	17	1	4	9
12	13	14	18	2	5	10
13	14	15	19	3	6	11
14	15	16	20	4	7	12
15	16	17	21	5	8	13
16	17	18	22	6	9	14
17	18	19	1	7	10	15
18	19	20	2	8	11	16
19	20	21	3	9	12	17
20	21	22	4	10	13	18
21	22	1	5	11	14	19
22	2	3	4	10	15	19
1	3	4	5	11	16	20
2	4	5	6	12	17	21
3	5	6	7	13	18	22
4	6	7	8	14	19	1
5	7	8	9	15	20	2
6	8	9	10	16	21	3
7	9	10	11	17	22	4
8	10	11	12	18	1	5
9	11	12	13	19	2	6
10	12	13	14	20	3	7
11	13	14	15	21	4	8
12	14	15	16	22	5	9
13	15	16	17	1	6	10
14	16	17	18	2	7	11
15	17	18	19	3	8	12
16	18	19	20	4	9	13
17	19	20	21	5	10	14
18	20	21	22	6	11	15
19	21	22	1	7	12	16
20	22	1	2	8	13	17
21	1	2	3	9	14	18

R22 IS A GYD(25 , 50 , 4)

25	24	8	16
24	6	4	20
6	18	11	7
18	12	9	21
12	25	19	23
1	17	22	11
17	14	3	9
14	15	2	19
15	10	5	8
10	1	13	4
7	21	24	2
21	23	6	5
23	16	18	13
16	20	12	22
20	7	25	3
19	8	17	18
8	4	14	12
4	11	15	25
11	9	10	24
9	19	1	6
13	22	21	15
22	3	23	10
3	2	16	1
2	5	20	17
5	13	7	14
25	2	10	18
24	5	1	12
6	13	17	25
18	22	14	24
12	3	15	6
1	18	20	15
17	12	7	10
14	25	21	1
15	24	23	17
10	6	16	14
7	15	9	16
21	10	19	20
23	1	8	7
16	17	4	21
20	14	11	23
19	16	5	11
8	20	13	9
4	7	22	19
11	21	3	8
9	23	2	4
13	11	12	2
22	9	25	5
3	19	24	13
2	8	6	22
5	4	18	3

R24 IS A GYD(25 , 50 , 12)

2	4	6	8	10	12	14	16	18	20	22	24
5	11	18	4	1	25	15	20	12	7	3	6
13	9	12	11	17	24	10	7	25	21	2	18
22	19	25	9	14	6	1	21	24	23	5	12
3	8	24	19	15	18	17	23	6	16	13	25
18	3	14	22	20	10	23	11	15	9	24	17
12	2	15	3	7	1	16	9	10	19	6	14
25	5	10	2	21	17	20	19	1	8	18	15
24	13	1	5	23	14	7	8	17	4	12	10
6	22	17	13	16	15	21	4	14	11	25	1
15	6	23	24	9	20	4	2	16	5	17	21
10	18	16	6	19	7	11	5	20	13	14	23
1	12	20	18	8	21	9	13	7	22	15	16
17	25	7	12	4	23	19	22	21	3	10	20
14	24	21	25	11	16	8	3	23	2	1	7
16	14	4	17	5	9	3	18	11	12	21	8
20	15	11	14	13	19	2	12	9	25	23	4
7	10	9	15	22	8	5	25	19	24	16	11
21	1	19	10	3	4	13	24	8	6	20	9
23	17	8	1	2	11	22	6	4	18	7	19
11	23	3	21	12	5	6	15	2	10	8	22
9	16	2	23	25	13	18	10	5	1	4	3
19	20	5	16	24	22	12	1	13	17	11	2
8	7	13	20	6	3	25	17	22	14	9	5
4	21	22	7	18	2	24	14	3	15	19	13
1	3	5	7	9	11	13	15	17	19	21	23
17	2	13	21	19	9	22	10	14	8	23	16
14	5	22	23	6	19	3	1	15	4	16	20
15	13	3	16	4	8	2	17	10	11	20	7
10	22	2	20	11	4	5	14	1	9	7	21
7	6	12	19	5	2	25	16	21	13	8	4
21	18	25	6	13	5	24	20	23	22	4	11
23	12	24	4	22	13	6	7	16	3	11	9
16	25	6	11	3	22	18	21	20	2	9	19
20	24	18	9	2	3	12	23	7	5	19	8
19	14	10	13	12	18	1	11	8	25	22	3
8	15	1	22	25	12	17	9	4	24	3	2
4	10	17	3	24	25	14	19	11	6	2	5
11	1	14	2	6	24	15	8	9	18	5	13
9	17	15	5	18	6	10	4	19	12	13	22
13	23	20	25	10	15	7	2	22	1	24	6
22	16	7	24	1	10	21	5	3	17	6	18
3	20	21	6	17	1	23	13	2	14	18	12
2	7	23	18	14	17	16	22	5	15	12	25
5	21	16	12	15	14	20	3	13	10	25	24
25	4	9	1	20	16	19	18	24	7	17	14
24	11	19	17	7	20	8	12	6	21	14	15
6	9	8	14	21	7	4	25	18	23	15	10
18	19	4	15	23	21	11	24	12	16	10	1
12	8	11	10	16	23	9	6	25	20	1	17

R21 IS A GYD(23 , 23 , 11)

1	4	9	16	2	13	3	18	12	8	6
2	5	10	17	3	14	4	19	13	9	7
3	6	11	18	4	15	5	20	14	10	8
4	7	12	19	5	16	6	21	15	11	9
5	8	13	20	6	17	7	22	16	12	10
6	9	14	21	7	18	8	23	17	13	11
7	10	15	22	8	19	9	1	18	14	12
8	11	16	23	9	20	10	2	19	15	13
9	12	17	1	10	21	11	3	20	16	14
10	13	18	2	11	22	12	4	21	17	15
11	14	19	3	12	23	13	5	22	18	16
12	15	20	4	13	1	14	6	23	19	17
13	16	21	5	14	2	15	7	1	20	18
14	17	22	6	15	3	16	8	2	21	19
15	18	23	7	16	4	17	9	3	22	20
16	19	1	8	17	5	18	10	4	23	21
17	20	2	9	18	6	19	11	5	1	22
18	21	3	10	19	7	20	12	6	2	23
19	22	4	11	20	8	21	13	7	3	1
20	23	5	12	21	9	22	14	8	4	2
21	1	6	13	22	10	23	15	9	5	3
22	2	7	14	23	11	1	16	10	6	4
23	3	8	15	1	12	2	17	11	7	5

R23 IS A GYD(25 , 25 , 9)

1	2	3	4	5	6	7	8	9
2	8	10	5	19	16	14	12	21
3	7	15	13	10	5	22	16	25
4	13	24	3	8	10	21	23	18
5	4	25	21	23	17	19	15	6
6	17	20	10	14	13	9	3	19
7	18	13	12	4	1	17	19	16
8	20	18	11	17	2	13	5	15
9	25	16	23	2	14	11	4	13
10	12	23	25	7	18	20	6	2
11	19	21	24	13	7	6	2	22
12	22	17	2	24	15	4	9	10
13	15	6	14	25	12	8	1	24
14	24	5	17	12	11	3	7	23
15	21	12	6	11	9	16	18	3
16	9	7	15	20	19	23	24	8
17	1	2	16	21	3	24	25	20
18	3	19	1	15	23	2	22	14
19	11	4	8	3	22	25	20	12
20	10	14	7	1	4	15	21	11
21	14	8	9	22	25	18	17	7
22	23	1	20	9	21	12	13	5
23	16	11	22	6	8	1	10	17
24	6	22	18	16	20	5	14	4
25	5	9	19	18	24	10	11	1

4.3. Listings for nonregular designs ($v \le 25$; $0 \ne c_1 \le c_2$).

For GYD(v, $b_1 = m_1 v + c_1$, $b_2 = m_2 v + c_2$) to exist v must divide $c_1 c_2$, and the b_i's, in addition to satisfying congruency to c_i mod v, must have, for $i = 1,2$ and $j = 2,1$ respectively, b_i is a multiple of

$$b_i \equiv \frac{v}{\text{g.c.d.}(v,c_j)} \cdot \frac{(v-1)}{\text{g.c.d.}(v-1,c_j(c_j-1))},$$

where g.c.d. denotes greatest common divisor.

Core designs G_0 = GYD(v,β_1,β_2) in this family are listed below, whenever these smallest possible constructions have $v \le 25$, β_1 and $\beta_2 \le 50$. Larger designs in the same family can be constructed by making a patchwork of regular designs, using B_1^0 = GYD(v,b_1^0,c_2) and B_2^0 = GYD(v,b_2^0,c_1) in the construction. (See 3.3.)

Table 2:
Small, core nonregular designs.

Listing Name	(v,c_1,c_2)-family	Core Design, G_0 (v,β_1,β_2)	Associated Designs $B_1^0 = (v,b_1^0,c_2)$	$B_2^0 = (v,b_2^0,c_1)$
N1	(4,2,2)	(4,6,6)	(4,12,2)	(4,12,2)
N2	(6,2,3)	(6,20,15)	(6,30,3)	(6,30,2)
N3	(6,3,4)	(6,15,10)	(6,30,4)	(6,30,3)
N4	(8,2,4)	(8,42,28)	(8,56,4)	(8,56,2)
N5	(8,4,4)	(8,28,28)	(8,56,4)	(8,56,4)
N6	(8,4,6)	(8,28,14)	(8,56,6)	(8,56,4)
N7	(9,3,3)	(9,12,12)	(9,36,3)	(9,36,3)
N8	(9,3,6)	(9,12,24)	(9,36,6)	(9,36,3)
N9	(9,6,6)	(9,24,24)	(9,36,6)	(9,36,6)
N10	(10,5,6)	(10,15,36)	(10,30,6)	(10,90,5)
N11	(10,5,8)	(10,45,18)	(10,90,8)	(10,90,5)
N12	(12,8,9)	(12,44,33)	(12,132,9)	(12,132,8)
– [4]	(15,5,6)	(15,35,21)	(15,105,6)	(15,105,5)
N14	(15,5,12)	(15,35,42)	(15,105,12)	(15,105,5)
N15	(16,4,4)	(16,20,20)	(16,80,4)	(16,80,4)
N16	(21,9,14)	(21,30,35)	(21,210,14)	(21,150,9)
N17	(25,5,5)	(25,30,30)	(25,150,5)	(25,150,5)
P18[5]	(25,40,40)	(25,40,40)	(25,300,15)	(25,300,15)

[4] No small design in this family exists, because of the non-existence of required BIB(15,21,5). See for example (Collens, 1976).

[5] The construction listed is a Pseudo Youden Design. See the Introduction.

N1 IS A GYD(4 , 6 , 6) .

Latin square
of order 4

2	4
3	2
4	1
1	3

3	1	4	2	1	2
4	2	1	3	3	4

N2 IS A GYD(6 , 15 , 20)

2×3 array of Latin squares
of order 6

5	1
5	6
4	1
4	6
3	5
3	4
3	1
3	6
2	6
2	5
2	4
2	1

2	3	1	6	4	3	5	2	3	3	2	6	1	5	2	4	3	2	5	4
5	6	4	5	1	1	6	4	4	4	1	5	4	6	6	1	5	5	1	6
3	2	3	1	2	6	4	6	5	2	3	2	6	3	1	5	1	4	2	3

N3 IS A GYD(6 , 10 , 15) .

1×2 array of Latin squares
of order 6

3	4	2
2	3	1
4	1	5
1	6	4
6	5	3
5	2	6

3	5	1	6	5	4	3	2	1	6	4	2	1	5	3
6	3	2	1	2	5	1	4	5	4	6	3	5	4	2
2	1	5	4	6	1	4	3	6	3	2	5	6	2	1
1	2	4	2	1	3	6	5	4	5	3	6	3	6	4

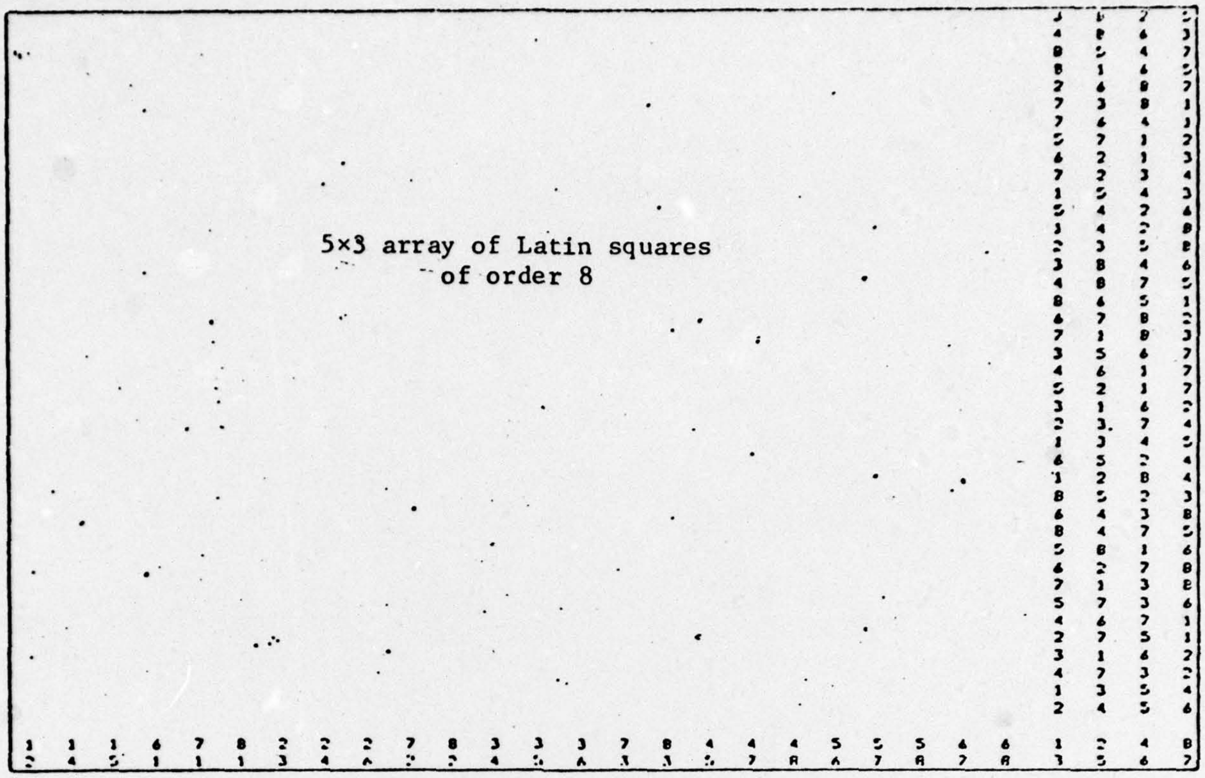
N4 IS A GYD(8 , 28 , 28)

3×3 array of Latin squares
of order 8

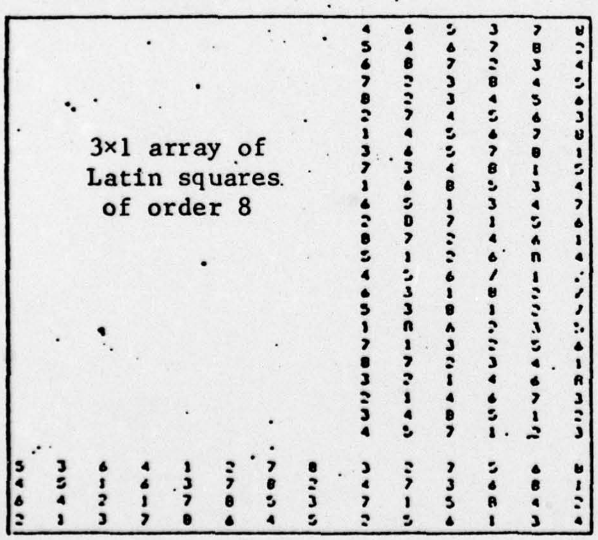
8	3	2	3
6	8	4	2
8	5	4	7
6	5	8	3
8	6	7	2
1	3	8	7
4	7	6	1
5	7	1	2
1	6	2	3
7	2	3	4
3	1	4	5
2	4	5	6
3	8	2	5
4	3	8	6
4	8	5	7
5	1	6	8
2	6	7	8
3	1	7	8
7	4	6	[illegible]
5	7	1	2
6	2	1	3
7	2	3	4
1	5	3	4
2	4	5	6

8	5	8	5	8	7	3	4	7	6	2	1	2	3	3	4	1	2	6	4	5	6	7	1	8	2	4	1
2	8	4	4	5	2	6	6	5	1	7	3	8	2	8	7	5	7	3	5	1	1	4	3	3	5	6	7
1	3	3	8	6	8	5	7	1	2	3	4	1	6	4	5	6	6	5	7	7	2	2	4	1	4	2	8
4	2	6	7	1	6	7	1	2	3	4	5	4	5	6	8	8	8	7	1	2	3	3	5	7	6	5	3

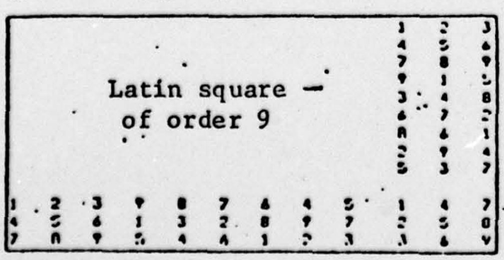
N5 IS A GYD(8 , 42 , 20)



N6 IS A GYD(8 , 28 , 14)



N7 IS A GYD(9 , 12 , 12)



N8 IS A GYD(9 , 12 , 24)

1×2 array of Latin squares of order 9

																		4	8	9	2	6	1
																		3	7	5	4	2	9
																		5	6	7	8	1	3
																		6	2	3	7	8	4
																		9	3	8	1	4	5
																		7	5	1	6	9	2
																		8	9	2	3	5	6
																		1	4	6	9	3	7
																		2	1	4	5	7	8
3	5	7	1	6	8	2	4	9	1	5	9	2	6	7	3	4	8	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	3	5	7	1	6	8	2	4	9	7	8	9	1	2	3
1	5	9	2	6	7	3	4	8	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6

N9 IS A GYD(9 , 24 , 24)

2×2 array of Latin squares of order 9

																		4	2	6	9	8	1
																		3	4	2	9	7	5
																		5	6	3	1	8	7
																		6	7	8	3	2	4
																		9	8	1	4	5	3
																		7	2	5	1	6	9
																		5	8	9	3	6	2
																		1	3	7	6	4	9
																		2	5	4	7	1	8
																		2	4	6	8	9	1
																		4	3	5	7	9	2
																		6	5	7	8	1	3
																		3	6	8	2	7	4
																		8	1	9	4	3	5
																		7	9	1	5	2	6
																		8	9	2	5	3	6
																		9	1	3	6	4	7
																		1	7	4	2	5	8
4	3	1	7	9	5	6	3	2	2	4	5	6	8	7	8	9	1	4	5	6	7	8	9
2	4	6	3	8	7	8	9	5	4	3	6	7	1	5	9	1	2	7	8	9	1	2	3
9	7	5	8	1	6	4	2	3	6	5	7	8	9	1	4	2	3	1	2	3	4	5	6
6	9	7	4	3	1	5	1	6	8	2	8	2	5	9	7	3	4	5	6	4	8	9	7
8	5	8	2	4	2	9	7	1	1	9	3	4	3	6	6	7	5	8	9	7	2	3	1
1	2	3	6	5	9	7	8	4	9	7	1	3	4	2	5	8	6	2	3	1	5	6	4

N10 IS A GYD(10 , 15 , 36)

1×3 array of Latin squares of order 10

																														10	2	5	6	3	1
																														4	1	2	9	7	5
																														1	7	6	3	8	4
																														9	10	3	8	4	2
																														8	4	9	5	1	6
																														7	8	10	1	5	3
																														6	9	1	10	2	7
																														5	3	8	2	6	9
																														2	5	7	4	10	8
																														3	6	4	7	9	10
5	10	9	4	9	5	6	10	4	3	1	7	3	2	8	6	6	5	1	3	8	7	2	1	9	7	10	4	8	2	5	6	7	4	3	2
7	8	2	9	5	10	1	2	3	9	4	1	6	10	5	10	2	8	3	6	7	4	8	9	3	1	4	5	7	6	1	7	2	3	9	8
6	1	3	3	4	4	2	7	10	1	9	5	8	3	10	2	7	1	9	8	5	8	10	7	6	4	9	2	6	5	6	8	1	10	2	4
8	9	4	10	2	2	7	8	8	5	6	3	1	5	6	5	4	3	10	1	1	10	7	2	4	3	6	9	9	7	9	3	4	5	10	1
4	5	6	7	8	6	10	9	1	2	10	4	2	7	3	1	2	9	7	2	6	5	3	4	8	5	1	8	10	9	7	9	8	6	5	10

N11 IS A GYD(10 , 45 , 18)

4×1 array of Latin squares of order 10

										10	2	6	8	7	5	9	4
										5	3	7	9	6	10	8	2
										3	5	7	6	4	9	10	8
										7	3	9	10	5	2	8	4
										5	6	2	3	10	9	4	8
										9	6	2	4	7	5	10	3
										4	8	3	5	2	6	7	10
										2	9	4	3	7	5	6	8
										1	10	5	7	9	8	6	4
										10	3	6	5	1	8	9	7
										3	4	6	7	8	9	10	1
										3	4	5	7	8	9	10	1
										4	5	6	8	9	10	1	3
										6	5	10	7	4	1	3	9
										7	6	8	10	1	3	4	5
										7	8	9	10	1	2	5	6
										9	8	2	1	10	4	6	7
										8	10	1	5	4	7	2	9
										9	7	4	6	5	2	1	10
										2	1	4	5	6	7	8	10
										2	4	5	6	7	8	9	1
										6	1	7	8	9	10	3	2
										5	7	8	9	10	1	2	3
										5	6	8	9	3	1	2	10
										6	7	8	10	5	2	3	1
										7	8	9	1	6	3	5	2
										8	9	10	1	4	3	2	7
										9	10	1	2	8	4	6	3
										10	1	4	3	2	6	7	9
										1	2	3	4	10	7	8	6
										2	3	7	6	1	8	9	4
										3	4	5	8	2	10	1	9
										4	5	10	9	3	7	1	2
										4	7	9	2	8	1	3	5
										6	9	10	1	2	3	4	5
										8	10	1	2	3	4	5	6
										8	9	1	2	3	4	5	6
										10	1	2	3	6	5	4	7
										1	2	3	4	9	6	7	5
										1	2	3	4	5	6	7	8
1	5	7	9	10	4	3	6	2	8	3	1	4	2	5	10	7	8
6	7	1	2	8	3	9	4	10	5	6	5	7	9	10	2	3	1
9	8	2	4	7	10	5	1	3	6	7	8	9	4	6	1	5	3
7	6	4	3	2	8	1	9	5	10	8	9	10	5	4	6	1	2
5	4	8	6	9	1	2	10	7	3	9	10	3	8	2	7	4	6

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N12 IS A GYD(12 , 44 , 33)

3×2 array of Latin squares of order 12

7	11	10	6	8	9	4	12	1
6	3	11	10	12	8	9	1	2
4	12	9	11	1	7	10	2	3
8	1	10	12	7	5	11	3	4
6	2	11	3	1	9	12	4	5
12	3	10	2	4	1	7	5	6
11	4	1	8	3	5	2	6	7
2	6	5	12	9	4	3	7	8
10	5	4	3	7	1	6	8	9
1	5	2	4	8	3	12	9	10
3	5	12	4	7	2	6	10	11
1	7	4	5	6	3	8	11	12
2	8	5	7	9	6	4	12	1
5	8	6	9	10	7	3	1	2
4	10	7	8	6	11	9	2	3
5	9	8	10	12	11	7	3	4
8	10	6	1	11	12	9	4	5
7	12	9	11	2	10	1	5	6
9	3	12	5	2	1	11	6	7
9	4	1	11	5	3	2	7	8
10	2	5	6	12	4	3	8	9
11	6	3	1	7	4	5	9	10
12	7	4	5	6	2	8	10	11
3	1	11	7	9	4	5	11	12
2	9	6	8	4	10	7	12	1
3	8	7	9	11	5	10	1	2
4	10	11	9	8	6	12	2	3
5	11	7	2	10	8	6	3	4
6	9	8	7	10	12	1	4	5
7	1	9	2	11	8	10	5	6
8	12	2	10	3	9	11	6	7
9	11	3	12	1	10	4	7	8
10	4	12	1	5	11	2	8	9
11	2	1	6	3	12	5	9	10
12	6	2	3	4	7	1	10	11
1	7	3	4	5	2	8	11	12

6	3	4	5	10	7	8	9	2	11	12	1	2	3	4	5	6	7	8	9	10	11	12	1	11	9	6	12	3	7	8	10	5
7	4	8	6	7	11	12	3	10	1	1	2	3	4	5	6	7	8	9	10	11	12	5	2	12	8	1	7	11	9	2	3	4
11	7	10	9	12	3	9	4	1	6	2	3	8	5	6	7	11	5	10	4	12	1	2	8	10	1	2	8	4	12	11	5	6
8	9	12	11	2	2	4	10	7	5	7	6	9	8	11	10	12	3	1	5	3	4	1	6	7	5	3	6	9	10	1	4	2
4	10	5	12	1	8	6	1	8	12	9	7	4	9	10	11	2	2	3	11	5	6	7	3	5	12	8	9	2	11	10	1	7
10	11	11	1	6	1	2	5	3	4	5	8	7	10	9	12	3	12	4	2	6	7	8	9	3	2	9	1	10	5	4	6	11
12	1	2	3	4	5	3	7	5	8	6	10	11	12	1	2	4	9	6	7	8	9	10	11	6	7	10	2	8	4	3	12	1
1	2	3	4	5	6	7	8	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12	9	11	12	4	6	3	5	7	8

N13 WOULD BE GYD(15,35,21), BUT DOESN'T EXIST
SINCE BIB(15,35,6,14,5) DOESN'T EXIST.

N14 IS A GYD(15 , 35 , 42)

2×2 array of Latin squares of order 15

12	5	2	3	4	11	15	1	9	14	6	8
4	1	11	13	6	2	7	8	15	14	10	3
6	10	2	12	9	7	5	14	1	15	13	4
3	8	15	1	12	14	9	11	5	10	13	7
3	11	5	10	7	13	12	8	9	4	2	6
10	4	15	11	13	1	2	3	7	12	5	9
8	13	1	14	3	6	5	2	10	7	9	12
11	12	13	7	14	8	15	6	5	4	1	2
8	11	14	13	6	15	3	9	1	10	4	12
10	14	3	4	5	9	6	7	11	15	2	8
11	12	6	1	5	3	4	2	10	9	14	13
2	10	12	7	14	8	6	15	3	1	11	5
14	7	8	15	13	12	11	10	2	9	1	4
1	3	7	15	9	6	10	11	13	8	4	5
2	15	14	9	12	13	8	5	4	6	3	7
5	2	9	8	7	15	14	4	13	3	11	1
12	15	10	6	1	2	4	13	8	7	3	9
9	14	1	5	2	11	12	10	6	13	8	15
7	6	8	9	10	3	1	4	14	5	12	11
13	3	11	5	4	7	10	6	12	2	15	14
1	9	10	3	2	4	11	15	7	12	6	14
13	1	5	2	11	9	3	12	15	8	7	6
5	7	9	2	8	4	13	1	6	11	14	10
6	8	7	4	15	1	14	13	3	5	12	10
15	2	4	10	8	5	13	12	14	11	9	3
15	6	13	14	3	10	8	9	4	2	5	1
14	5	4	11	10	12	1	3	6	13	7	2
4	9	12	6	15	10	7	5	2	1	8	11
7	4	3	12	1	5	9	14	11	6	15	13
9	13	6	8	11	14	2	7	12	3	10	15

3	9	8	5	4	6	14	13	7	1	12	11	15	1	2	9	10	3	8	7	10	2	5	4	6	11	15	14	13	12	12	11	6	9	8	4	10	5	7	13	14	15
2	1	5	12	3	15	4	9	8	10	13	9	11	7	6	6	4	7	14	10	2	1	8	5	3	14	13	12	11	15	14	13	9	10	6	7	15	11	3	2	5	1
5	4	1	2	6	10	9	8	12	11	10	3	8	15	5	13	12	9	6	14	11	13	14	7	15	7	4	3	2	1	13	12	11	2	3	14	7	6	1	8	9	4
15	14	3	7	11	9	10	7	2	6	1	12	4	3	13	5	8	11	2	1	14	15	12	13	4	5	6	10	9	8	15	14	1	7	10	9	8	2	12	3	4	5
10	2	13	4	1	5	8	3	6	7	4	5	2	14	14	12	11	15	15	13	3	9	1	11	12	8	7	6	10	9	11	15	8	12	13	6	5	1	2	4	3	10

N15 IS A GYD(16 , 20 , 20)

Latin square of order 16

																9	13	5	1
																10	14	6	2
																11	15	7	3
																12	16	8	4
																16	7	2	9
																13	6	3	12
																14	5	4	11
																15	8	1	10
																2	11	13	8
																3	10	16	5
																1	12	14	7
																4	9	15	6
																5	2	12	15
																8	3	9	14
																7	4	10	13
																6	1	11	16
3	6	12	16	2	13	14	10	11	5	7	9	15	8	4	1	1	2	3	4
2	8	9	15	16	12	4	1	13	10	14	6	5	3	7	11	5	6	7	8
4	5	10	14	7	3	11	8	2	16	1	15	12	9	13	6	9	10	11	12
1	7	11	13	9	6	5	15	8	3	12	4	2	14	10	16	13	14	15	16

N16 IS A GYD(21 , 30 , 35)

Latin square of order 21

																					2	4	6	12	8	3	10	15	16	17	11	20	14	21
																					3	5	4	6	7	12	17	9	20	11	10	21	15	1
																					20	7	5	8	9	10	21	18	2	12	19	13	11	4
																					10	9	11	7	21	2	13	14	19	20	16	15	1	3
																					17	16	9	10	3	11	7	6	13	14	21	1	4	5
																					18	10	8	17	12	21	14	1	7	15	13	4	6	2
																					8	21	14	11	18	15	19	2	3	6	1	5	13	7
																					6	11	10	21	14	16	5	12	1	4	2	8	19	9
																					7	1	12	15	13	17	11	16	8	2	3	9	20	6
																					9	12	13	14	10	1	8	3	11	18	20	2	5	17
																					12	2	19	18	15	14	1	20	6	3	4	7	9	10
																					16	14	15	13	4	20	6	19	5	9	8	3	7	12
																					4	15	16	2	20	18	9	17	21	5	7	14	10	13
																					15	18	17	5	19	6	20	8	16	21	9	11	3	14
																					11	19	10	18	5	13	15	10	4	1	12	6	21	20
																					21	17	1	20	2	19	3	5	15	7	10	12	5	16
																					13	6	20	19	17	7	4	21	18	8	3	16	2	11
																					19	20	21	3	1	9	16	4	17	13	14	18	12	8
																					14	13	7	9	6	8	18	11	12	19	17	10	16	15
																					5	3	2	1	11	4	12	13	14	16	15	19	17	18
																					1	8	3	4	16	5	2	7	9	10	6	17	18	19
1	2	3	4	5	6	7	21	15	16	17	18	19	20	8	9	10	11	12	13	14	14	5	10	7	6	8	16	12	20	21	3	13	1	18
2	3	4	5	6	7	1	19	20	21	15	16	17	18	14	8	9	10	11	12	13	5	8	21	1	2	17	14	9	13	19	4	20	6	15
3	4	5	6	7	1	2	18	19	20	21	15	16	17	13	14	8	9	10	11	12	2	18	19	15	12	13	3	16	17	5	6	21	9	10
5	6	7	1	2	3	4	17	18	19	20	21	15	16	11	12	13	14	8	9	10	16	3	9	5	11	15	18	7	4	14	12	8	21	2
8	9	10	11	12	13	14	1	2	3	4	5	6	7	21	15	16	17	18	19	20	8	9	7	19	21	3	1	17	12	13	15	4	10	11
9	10	11	12	13	14	8	7	1	2	3	4	5	6	19	20	21	15	16	17	18	19	17	4	10	20	7	15	11	1	18	14	16	5	8
10	11	12	13	14	8	9	6	7	1	2	3	4	5	18	19	20	21	15	16	17	4	6	18	20	1	2	13	3	10	11	8	9	15	16
12	13	14	8	9	10	11	4	5	6	7	1	2	3	17	18	19	20	21	15	16	10	11	6	13	14	12	4	19	2	3	20	5	17	7
17	18	19	20	21	15	16	8	9	10	11	12	13	14	1	2	3	4	5	6	7	17	20	12	11	16	21	9	1	18	2	19	7	14	6

N17 IS A GYD(25 , 30 , 30)

Latin square of order 25

																									6	11	16	1	21
																									22	2	12	17	7
																									23	13	3	18	8
																									19	4	14	24	9
																									15	25	10	20	5
																									2	15	19	23	6
																									11	3	24	7	20
																									16	12	8	4	25
																									21	5	13	9	17
																									1	22	18	14	10
																									10	16	2	13	24
																									25	6	17	3	14
																									7	21	4	15	18
																									5	8	11	19	22
																									20	1	9	12	23
																									9	18	25	2	11
																									12	19	21	10	3
																									4	20	6	22	13
																									14	23	7	5	16
																									24	17	15	8	1
																									8	14	20	21	2
																									3	9	22	16	15
																									17	10	23	11	4
																									18	24	5	6	12
																									13	7	1	25	19
2	7	12	17	22	6	11	16	21	1	10	25	15	5	20	9	19	4	14	24	8	3	23	18	13	1	2	3	4	5
3	8	13	18	23	2	7	12	17	22	16	6	21	11	1	25	10	20	5	15	14	9	4	24	19	6	7	8	9	10
4	9	14	19	24	23	3	8	13	18	2	17	7	22	12	11	21	6	16	1	20	15	10	5	25	11	12	13	14	15
5	10	15	20	25	19	24	4	9	14	13	3	18	8	23	2	12	22	7	17	21	16	11	6	1	16	17	18	19	20
1	6	11	16	21	15	20	25	5	10	24	14	4	19	9	18	3	13	23	8	2	22	17	12	7	21	22	23	24	25

~~N18 WOULD BE GYD(25,40,40)? IT MAY OR MAY NOT EXIST.~~

SHUFFLE

```
80 PRINT'FILE NAMES?'
90 INPUT A$,B$
100 FILE #1:A$
110 FILE #2:B$
115 MARGIN #2:0
120 DIM A(50,25),N(50,25),G(50)
130 INPUT #1:V,B,K
140 MAT A=ZER(B,K)
150 MAT N=ZER(B,K)
160 MAT INPUT #1:A(B,K)
170 LET M=INT(B/V)
180 FOR P=1 TO V
190      FOR J=1 TO K
200           FOR I=1 TO B
210                IF A(I,J)=P
220                THEN LET A(I,J)=P+C*V
230                     LET N(P+C*V,J)=N(P+C*V,J)+1
240                     LET C =MOD(C+1,M)
250                CONTINUE
260           NEXT I
270      NEXT J
280      IF C>0
290      THEN PRINT 'IMPROPER # OF TREATMENTS';P
300           LET Q=1
310      CONTINUE
320 NEXT P
330
332 LET M=1
334 PERFORM CHECK
336 DO WHILE M=1
338    PERFORM CHECK
340    FOR C=1 TO K
350         FOR R=1 TO B
360              IF N(R,C)=0
370              THEN LET C1=1
380                   DO WHILE N(R,C1)<=1
390                      LET C1=C1+1
400                      IF C1>K THEN PRINT 'HELP'
410                   LOOP
420                   MAT G=ZER(B)
430                   LET X=R
440                   PERFORM FINDY
450                   DO WHILE N(Y,C)=1 AND N(Y,C1)>0
460                      PERFORM EXCHANGE
470                      LET X=Y
480                      PERFORM FINDY
490                   LOOP
500                   PERFORM EXCHANGE
510              CONTINUE
520         NEXT R
530    NEXT C
535 LOOP
```

```
540 IF Q=0
545        PERFORM UNSCRAMBLE
550 THEN
570        SCRATCH #2
575        PRINT #2:V;",";B;",";K
580        FOR I=1 TO B
590            FOR J=1 TO K
600                PRINT #2:A(I,J);",";
610            NEXT J
620            PRINT #2:
630        NEXT I
640 ELSE PRINT 'NO WRITING DONE IN FILE';B$
650 CONTINUE
670
680 DEFINE FINDY
690       LET I=1
700       DO WHILE A(I,C1)<>X OR G(I)=1
710          LET I=I+1
720          IF I>B THEN PRINT 'HELP'
730       LOOP
740       LET Y=A(I,C)
750       LET G(I)=1
760 DEFEND
770
780 DEFINE EXCHANGE
790       LET A(I,C)=X
800       LET A(I,C1)=Y
810       LET N(X,C)=N(X,C)+1
820       LET N(X,C1)=N(X,C1)-1
830       LET N(Y,C)=N(Y,C)-1
840       LET N(Y,C1)=N(Y,C1)+1
850 DEFEND
855
860 DEFINE CHECK
862       REM: RETURNS M=1 IF MATRIX NOT RIGHT
865       LET M=0
870       FOR C=1 TO K
880           FOR R=1 TO B
890               IF N(R,C)<>1 THEN LET M=1
900           NEXT R
910       NEXT C
920 DEFEND
925
930 DEFINE UNSCRAMBLE
932 FOR J=1 TO K
934       FOR I=1 TO B
936           LET A(I,J)=MOD(A(I,J)-1,V)+1
938       NEXT I
940 NEXT J
942 DEFEND
1000 END
READY
```

References

1. Agrawal, H. (1966). Some generalizations of distinct representatives with applications to statistical designs. Ann. Math. Statist. 37 525-528.

2. Cheng, C.-S. (1978). Optimality and Construction of Pseudo Youden Designs. Technical Report, Univ. of California, Berkeley, CA.

3. Collens, R. J. (1976). Constructing BIBD's with a computer. Ars Combinatoria 2 285-303.

4. Hanani, H. (1975). Balanced incomplete block designs and related designs. Discrete Math. 11 255-369.

5. Kiefer, J. (1975). Balanced block designs and generalized Youden designs, 1. Construction (patchwork). Ann. Statist. 3 109-118.

6. Kiefer, J. (1975a). Construction and Optimality of Generalized Youden Designs. In A Survey of Statistical Design and Linear Models (J. N. Srivastava, ed.) 333-353. North-Holland, Amsterdam.

7. Ruiz, F. and Seiden, E. (1974). On construction of some families of generalized Youden designs. Ann. Statist. 2 503-519.

8. Seiden, E. and Wu, C. Y. (1978). A geometric construction of generalized Youden designs for v a power of a prime. Ann. Statist. 6 452-460.

Generalized Youden Designs are generalizations of the class of two-way balanced block designs which include Latin squares and Youden squares. They are used for the same purposes and in the same way that these classical designs are used, and satisfy most of the most common criteria of design optimality.

We explicitly display or give detailed instructions for constructing all these designs within a practical range: when v, the number of treatments, is $\leq$ 25; and b_1 and b_2, the dimensions of the design array, are each $\leq$ 50.

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